Discrete Distributions

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Different (overdispersed) count data approaches

(i) *Ad-hoc* solutions
  (a) quasi-likelihood (QL), Extended QL
  (b) Efron's Double Exponential
  (c) pseudo-likelihood (PL)

(ii) Discretized continuous distributions
for example if $F_W(w)$ is the cdf a continuous random variable W defined in $\Re^+$ then $f_Y(y) = F_W(y+1) - F_W(y)$

(iii) Random effect at the observation level solutions.
$f_Y(y) = \int f(y|\gamma) f_\gamma(\gamma) d\gamma$.

Random effect at the observation level

(a) when an an explicit continuous mixture distribution, $f_Y(y)$, exists.

(b) when a continuous mixture distribution, $f_Y(y)$, is not explicit but is approximated by integrating out the random effect using approximations, e.g. Gaussian quadrature or Laplace approximation.

(c) when a 'non-parametric' mixture (effectively a finite mixture) is assumed for the response variable.

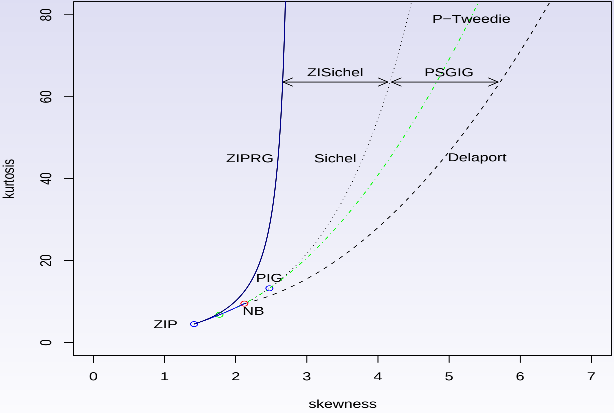
Explicit continuous mixture distribution

Distributions	R Name	mixing distribution for γ
Poisson	$\texttt{PO}(\mu)$	-
Neg. bin. I	$\texttt{NBI}(\mu, \sigma)$	$\texttt{GA}(1, \sigma^{\frac{1}{2}})$
Neg. bin. II	$\texttt{NBII}(\mu, \sigma)$	$\texttt{GA}(1, \sigma^{\frac{1}{2}}/\mu)$
Poisson IG	$\texttt{PIG}(\mu, \sigma)$	$\texttt{IG}(1, \sigma^{\frac{1}{2}})$
Sichel	$\texttt{SICHEL}(\mu, \sigma, \nu)$	$\texttt{GIG}(1, \sigma^{\frac{1}{2}}, \nu)$
Delaporte	$\texttt{DEL}(\mu, \sigma, \nu)$	$\texttt{SG}(1, \sigma^{\frac{1}{2}}, \nu)$
Zero inflated Poisson	$\texttt{ZIP}(\mu, \sigma)$	$\texttt{BI}(1, 1-\sigma)$
Zero inflated Poisson 2	$\texttt{ZIP2}(\mu, \sigma)$	$(1-\sigma)^{-1}\texttt{BI}(1, 1-\sigma)$
Zero inflated neg. bin.	$\texttt{ZINBI}(\mu, \sigma, \nu)$	zero inflated gamma
Poisson-Tweedie	-	Tweedie family

Table : Discrete gamlss family distributions for count data

R Name	params	mean	variance
$\texttt{PO}(\mu)$	1	μ	μ
$\texttt{NBI}(\mu, \sigma)$	2	μ	$\mu + \sigma\mu^2$
$\texttt{NBII}(\mu, \sigma)$	2	μ	$\mu + \sigma\mu$
$\texttt{PIG}(\mu, \sigma)$	2	μ	$\mu + \sigma\mu^2$
$\texttt{SICHEL}(\mu, \sigma, \nu)$	3	μ	$\mu + h(\sigma, \nu)\mu^2$
$\texttt{DEL}(\mu, \sigma, \nu)$	3	μ	$\mu + \sigma(1-\nu)^2\mu^2$
$\texttt{ZIP}(\mu, \sigma)$	2	$(1-\sigma)\mu$	$(1-\sigma)\mu + \sigma(1-\sigma)\mu^2$
$\texttt{ZIP2}(\mu, \sigma)$	2	μ	$\mu + \frac{\sigma}{(1-\sigma)}\mu^2$

Comparison of the marginal distributions using a (ratio moment) diagram of their skewness and kurtosis



gamlss

Families modelling the variance-mean relationship

$V[Y] = \mu + \mu^2 V[\gamma]$ where $V[\gamma] = \upsilon(\sigma, \nu, \tau)$ is a function of the parameters of the mixing distribution $f_\gamma(\gamma)$.
Alternative variance-mean relationship can be obtained by reparametrization.
i.e NB type I $V[Y] = \mu + \sigma\mu^2$.
If $\sigma = \sigma_1/\mu$ then
$V[Y] = (1 + \sigma_1)\mu$ (negative binomial type II)
$\sigma = \sigma_1\mu$ then $V[Y] = \mu + \sigma_1\mu^3$.
More generally $\sigma = \sigma_1\mu^{2-\nu}$ giving $V(Y) = \mu + \sigma_1\mu^\nu$

A stylometric application

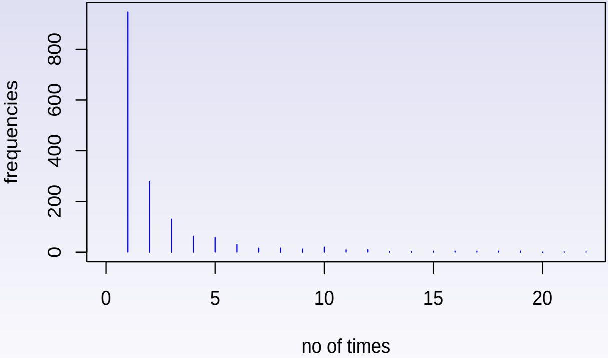
Data summary:

R data file:	`stylo` in package **gamlss.data** of dimensions 64 × 2
source:	Dr Mario Corina-Borja
variables	
	`word` : is the number of times a word appears in a single text
	`freq` : the frequency of the number of times a word appears in a text
purpose:	to demonstrate the fitting of a truncated discrete dist.
conclusion	the truncated SICHEL distributions fits best

A stylometric application

```
 library(gamlss.tr)
 data(stylo)
 plot(freq ~ word, data = stylo, type = "h", xlim =
+  c(0, 22), xlab = "no of times", ylab =
+ "frequencies", col = "blue")
```

The stylometric data



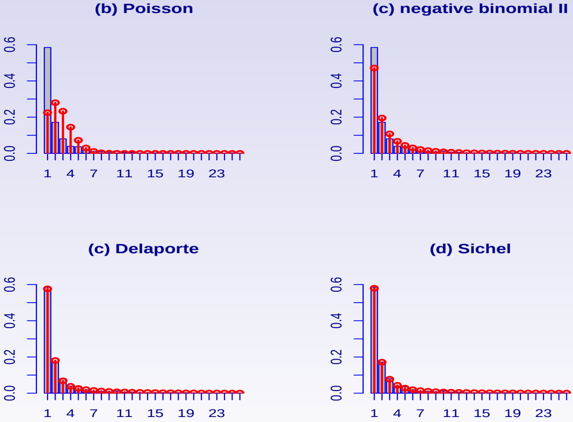
A stylometric application

```
> library(gamlss.tr)
> gen.trun(par = 0, family = PO, type = "left")
A truncated family of distributions from PO has been genera
 and saved under the names:
 dPOtr pPOtr qPOtr rPOtr POtr
The type of truncation is left and the truncation parameter
> gen.trun(par = 0, family = NBII, type = "left")
...
> gen.trun(par = 0, family = DEL, type = "left")
...
> gen.trun(par = 0, family = SICHEL, type = "left",
+      delta = 0.001)
...
```

A stylometric application

```
> mPO <- gamlss(word ~ 1, weights = freq, data = stylo,
+ family = POtr, trace = FALSE)
> mNBII <- gamlss(word ~ 1, weights = freq, data = stylo,
+ family = NBIItr, n.cyc = 50, trace = FALSE)
> mDEL <- gamlss(word ~ 1, weights = freq, data = stylo,
+ family = DELtr, n.cyc = 50, trace = FALSE)
> mSI <- gamlss(word ~ 1, weights = freq, data = stylo,
+ family = SICHELtr, n.cyc = 50, trace = FALSE)
> GAIC(mPO, mNBII, mDEL, mSI)
      df      AIC
mSI    3 5148.454
mDEL   3 5160.581
mNBII  2 5311.627
mPO    1 9207.459
```

The stylometric data



The fish species data

Data summary: the fish species data

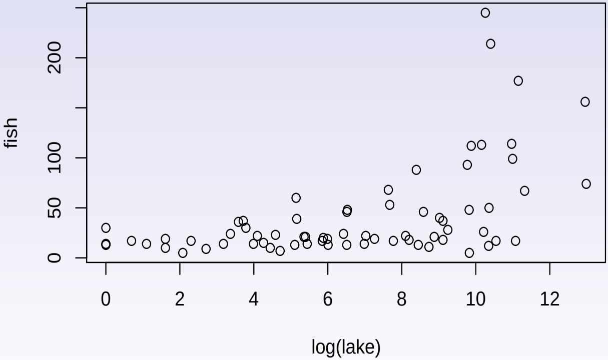
R data file: `species` in package gamlss.data of dimensions 70 × 2

variables

`fish` : the number of different species in 70 lakes in the world

`lake` : the lake area

The fish species data



The fish species data

There are several questions that need to be answered.

- How does the mean of y depend on x?
- Is y overdispersed Poisson?
- How does the variance y depend on its mean?
- What is the distribution of y given x?
- Do the scale and shape parameters of the distribution of y depend on x?

Overdispersed count data approaches

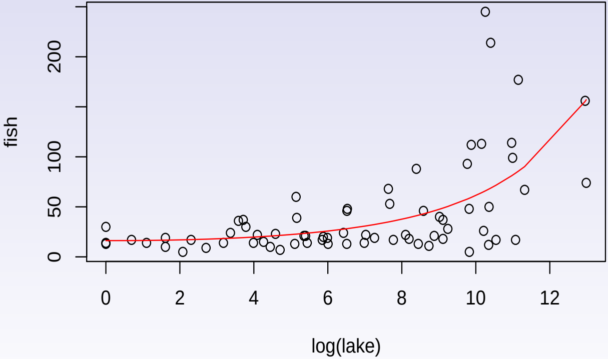
Table : Comparison of models for the fish species data

Model	$f_Y(y)$	μ	σ	ν	DEV	df	AIC	SBC
1	PO	$x < 2 >$	-	-	1849.3	3	1855.3	1862.0
2	NBI	x	1	-	619.8	3	625.8	632.6
3	NBI	$x < 2 >$	1	-	614.3	4	622.3	631.3
4	NBI	$cs(x, 3)$	1	-	611.9	6	623.9	637.4
5	NBI	$x < 2 >$	x	-	605.0	5	615.0	626.2
6	NBI-fam	$x < 2 >$	1	1	606.0	5	616.0	627.3
7	NBI-fam	$x < 2 >$	x	1	604.9	6	616.9	630.4

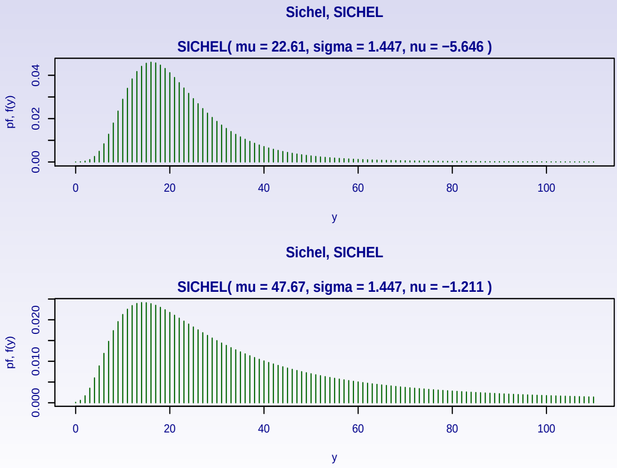
Overdispersed count data approaches

Model	$f_Y(y)$	μ	σ	ν	DEV	df	AIC	SBC
8	PIG	$x<2>$	1	-	613.3	4	621.3	630.3
9	SI	$x<2>$	1	x	597.7	6	609.7	623.2
10	DEL	$x<2>$	1	x	600.6	6	612.6	626.1
11	DEL	$x<2>$	-	x	600.6	5	610.6	621.9
12	PO-Normal	$x<2>$	1	-	615.2	4	623.2	632.2
13	NBI-Normal	$x<2>$	x	1	603.7	6	615.7	629.2
14	PO-NPFM(5)	$x<2>$	-	—	601.9	13	627.9	657.2
15	NB-NPFM(2)	$x<2>$	1	—	611.9	6	623.9	637.4
16	doublePO	$x<2>$	x	-	616.4	5	626.4	637.6
17	IGdisc	$x<2>$	1	-	603.3	4	611.3	620.3

Fitted mean of the Sichel distribution



Fitted Sichel distributions for observations (a) 40 and (b) 67



Binomial response variables

There are only two distributions here

- binomial
- beta binomial

The alveolar-bronchiolar adenomas data

Data summary:

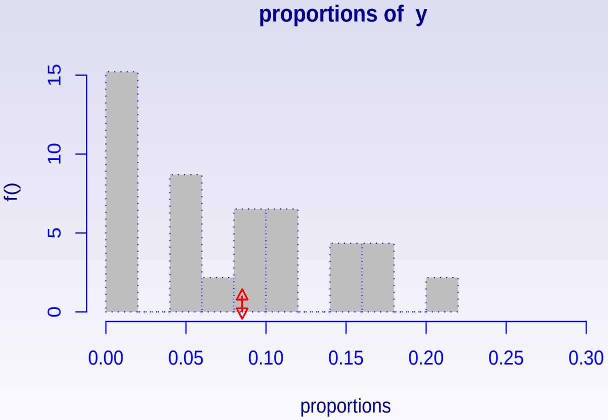
R data file:	`alveolar` in package **gamlss.data** of dimensions 23 × 2
source:	Tamura and Young (1987), and Hand *et al.* (1994)
variables	

`r` : number of mice having alveolar-bronchiolar
`n` : total number of mice

The alveolar-bronchiolar adenomas data

```
> data(alveolar)
> alveolar$y <- with(alveolar, cbind(r, n - r))
> m1 <- gamlss(y ~ 1, data = alveolar, family = BI)
> m2 <- gamlss(y ~ 1, data = alveolar, family = BB)
> GAIC(m1, m2)
df AIC
m1 1 73.1292
m2 2 75.0665
> m3 <- with(alveolar, histDist(y, "BI", xlim = c(0, 0.3))
```

The alveolar-bronchiolar adenomas data



The hospital stay data

Data summary:

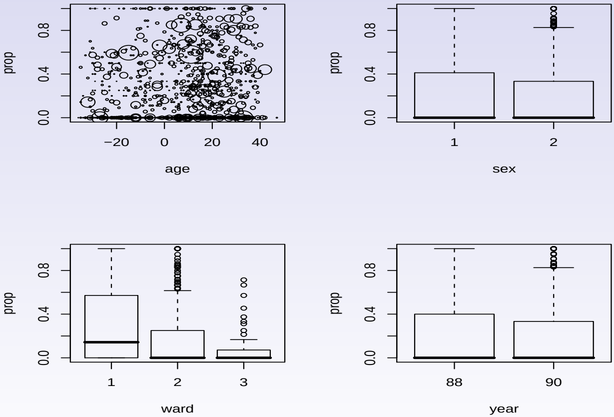
R data file: aep in package **gamlss** of dimensions 1383 × 8

source: Gange *et al.* (1996)

variables

`los` : total number of days
`noinap` : number of inappropriate days patient stay in hospital
`loglos` : the log of `los/10`
`sex` : the gender of patient
`ward` : type of ward in the hospital (medical, surgica
`year` : 1988 or 1990
`age` : age of the patient subtracted from 55
`y` : the response variable, a matrix with columns (`noinap`, `los-noinap`)

The hospital stay data



The hospital stay data

```
> mI <- gamlss(y ~ ward + year + loglos, sigma.fo = ~year,
+ family = BB,data = aep)
> mII <- gamlss(y ~ ward + year + loglos, sigma.fo = ~year
+ ward, family = BB, data = aep)
> mIII <- gamlss(y ~ ward + year + cs(loglos, 1),
+ sigma.fo = ~year+ward, family = BB, data = aep)
> mIV <- gamlss(y ~ ward + year + cs(loglos, 1) + cs(age, 1
+ sigma.fo = ~year + ward, family = BB, data = aep)
> GAIC(mI, mII, mIII, mIV, k = 0)
           df      AIC
mIV  12.00010 4454.362
mIII 10.00045 4459.427
mII   9.00000 4483.020
mI    7.00000 4519.441
```

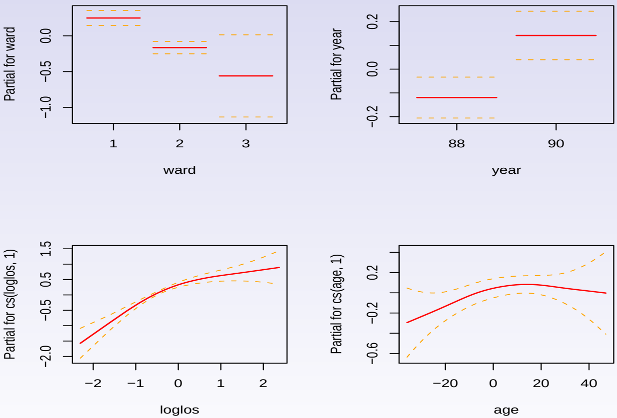
The hospital stay data

Models	Links	Terms	GD (AIC) [SBC]
I	$\mathrm{logit}(\mu)$ $\log(\sigma)$	1+ward+loglos+year 1+year	4519.4 (4533.4) [4570.1]
II	$\mathrm{logit}(\mu)$ $\log(\sigma)$	1+ward+loglos+year 1+year+ward	4483.0 (4501.0) [4548.1]
III	$\mathrm{logit}(\mu)$ $\log(\sigma)$	1+ward+cs(loglos,1)+year 1+year+ward	4459.4 (4479.4) [4531.8]
IV	$\mathrm{logit}(\mu)$ $\log(\sigma)$	1+ward+cs(loglos,1)+year+cs(age,1) 1+year+ward	4454.4 (4478.4) [4541.2]

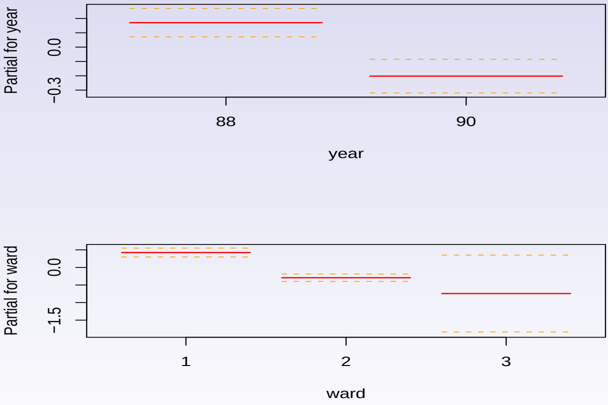
The hospital stay data

```
> op <- par(mfrow = c(2, 2))
> term.plot(mIV, se = T)
> par(op)
> op <- par(mfrow = c(2, 1))
> term.plot(mIV, "sigma", se = T)
> par(op)
> rqres.plot(mIV)
```

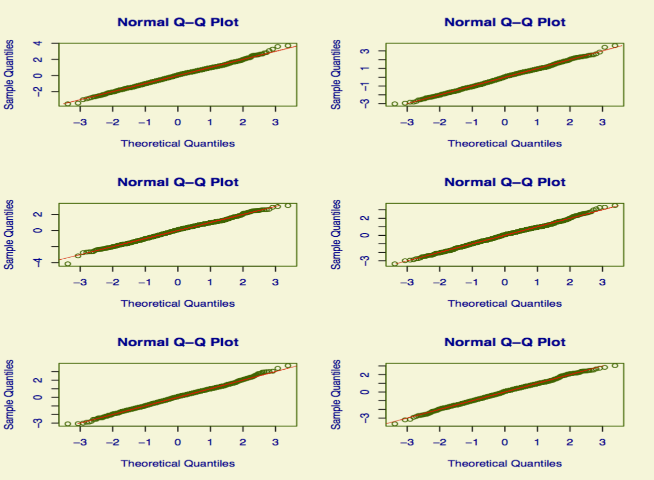
The hospital stay data



The hospital stay data



The hospital stay data



END

for more information see

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