Smoothing

Mikis Stasinopoulos[1] Bob Rigby[1]

[1]STORM, London Metropolitan University

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   - Weighed running means smoothers
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Health and Lifestyle Survey Data

5191 observations

Adults age 18 upwards, in England, Wales and Scotland, 1984-5.

Height: was measured without shoes using a stadiometer reading to 1 mm.

Income: Household Equivalent Income obtained using the McClemets scale which divides the net household income by a factor which is dependent on the presence of spouse, number of dependent children and the number of other adults in the household.

Health and Lifestyle Survey Data: plot

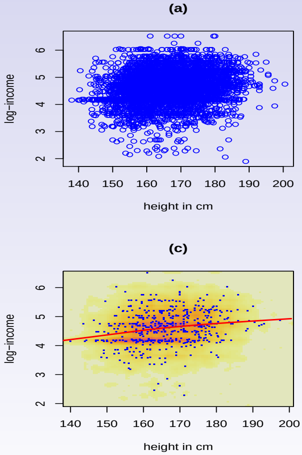

Figure: Height against log-income together with a lowess smoother

Reporting crime in the local media: the data

10590 observations

media: whether crime reported ($y = 1$) or not ($y = 0$) in the local media

age: the age of the victim of crime

Reporting crime in the local media: plot

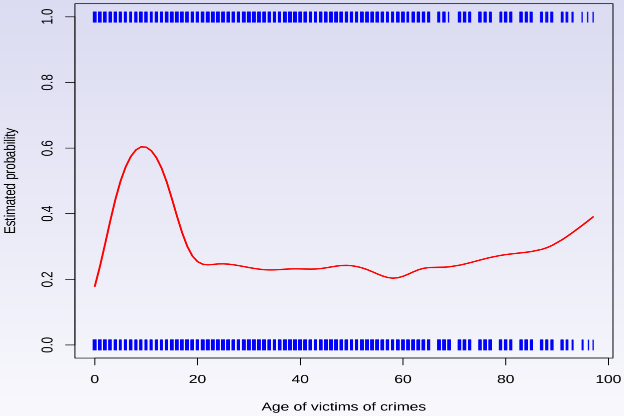

Figure: Reported cases by media against the age of the victim.

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Athens pollution data

deaths: daily total counts of deaths in Athens, Greece for the period 1st of January 1988 - 31st December 1991.

year: daily data

Purpose: to provide a quantitative summary of the short-term effects of specific air pollutants on mortality and morbidity.

Athens was one of the 30 European cities studied in the project.

To adjust for long term trend patterns on the daily counts of deaths.

This is the first step in order to access the short term air pollutant effect
(Katsougianni et al (1992)).

Athens pollution data: plot

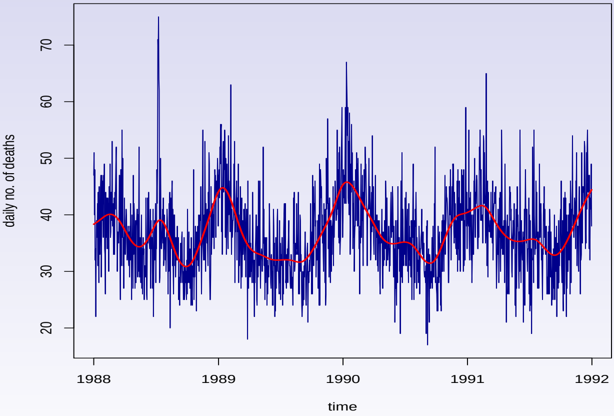

Figure: The number of daily deaths

Introduction

Scatter plot smoothers:

- local polynomials
- Penalised piecewise polynomials

Local polynomials

- Unweighted
  - Running means smoother
  - Local polynomial smoother
- Weighted
  - weighed running means smoother or kernel smoother
  - weighed local polynomials smoother
    - loess

Running means smoothers

demo.Locmean(): for symmetric local window mean fitting

Local polynomials smoother

demo.Locpoly(): for symmetric local window polynomial fitting

Weighed running means smoothers

demo.WLocmean(): for Gaussian kernel fitting

Weighed running means smoothers

demo.WLocpoly(): for weighed local polynomial fitting using a Gaussian kernel window

Weighed running means smoothers: summary

- weighted smoother better in the eye than unweighted
- smoothing parameter: span or λ.
- important how the x-values are taken into the account i.e. local symmetric window, weighted window etc.
- linear smoothers $\hat{\mathbf{y}} = \mathbf{S}\mathbf{y}$.
- smoother are effected differently with the sparseness of the x-values. i.e. kernel smoother can misbehave at the end of the data
- smoothers are robust for extreme values at x-axis since only local observations taking part on the fit.
- influential observations in the y-axis do effect the smoothers.

Penalised piecewise polynomial smoothers

- Piecewise Polynomials
- Regressions Splines (without penalty)
- Penalised Regression Splines

Polynomials basis

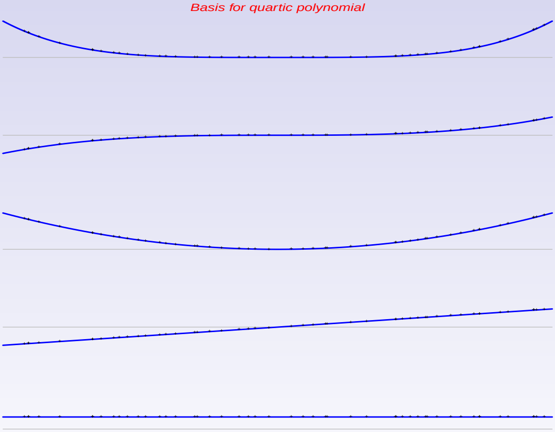

Figure: Polynomial Basis.

Orthogonal Polynomials basis

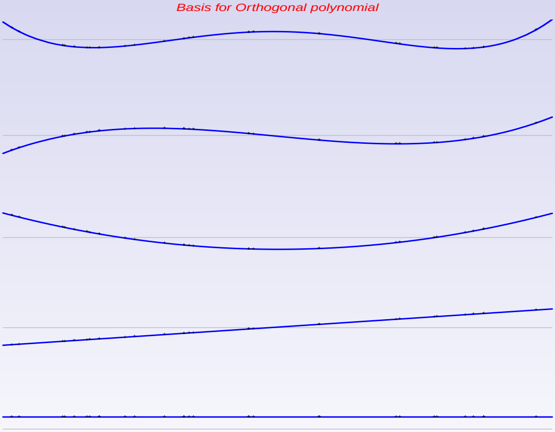

Figure: Polynomial Basis.

Piecewise Polynomials: linear

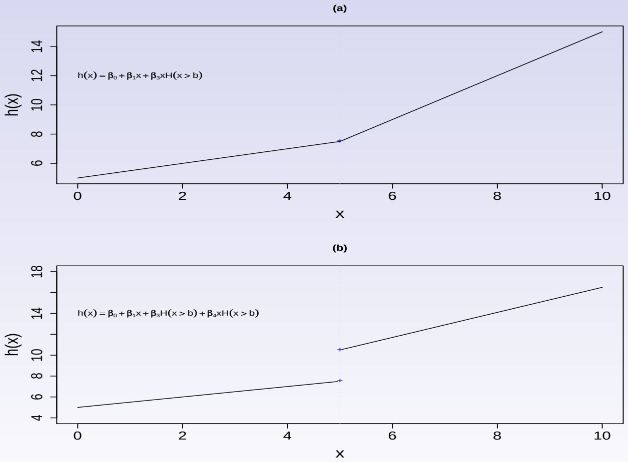

Figure: Piecewise linear, continuous(a) and discontinuous (b) lines.

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Piecewise Polynomials: linear

$$h(x) = \beta_{00} + \beta_{01}x + \beta_{11}(x - b)H(x > b) \qquad (1)$$

$$h(x) = \beta_{00} + \beta_{01}x + [\beta_{10} + \beta_{11}(x - b)]\, H(x > b) \qquad (2)$$

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Piecewise Polynomials: quadratic

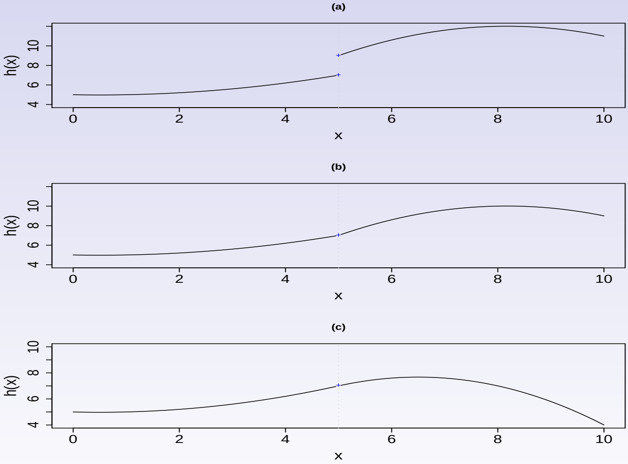

Figure: Piecewise quadratic.

Piecewise Polynomials: quadratic

$$
\begin{aligned}
h(x) &= \beta_{00} + \beta_{01}x + \beta_{02}x^2 + \\
&\quad \left[\beta_{10} + \beta_{11}(x-b) + \beta_{12}(x-b)^2\right] H(x > b)
\end{aligned}
$$

Piecewise Polynomials: general

Piecewise Polynomials:

$$h(x) = \sum_{j=0}^{D} \beta_{0j} x^j + \sum_{k=1}^{K} \sum_{j=0}^{D} \beta_{kj} (x - b_k)^j H(x > b_k) \qquad (3)$$

Splines

$$h(x) = \sum_{j=0}^{D} \beta_{0j} x^j + \sum_{k=1}^{K} \beta_k (x - b_k)^D H(x > b_k) \qquad (4)$$

Cubic Splines

$$h(x) = \beta_{00} + \beta_{01} x + \beta_{02} x^2 + \beta_{03} x^3 + \sum_{k=1}^{K} \beta_k (x - b_k)^3 H(x > b_k) \quad (5)$$

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Beta splines

demo.BetaSplines(): for B-spline

Beta splines

- B-splines are local functions, look like Gaussian
- B-splines are columns of basis matrix $\mathbf{B}$
- scalling a summing gives fitted values: $\boldsymbol{\mu} = \mathbf{B}\gamma$
- the knots determine the B-spline basis
- Polynomial pieces make up B-splines, joint at knots
- General patterns of knots are possible

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Regressions Splines (without penalty)

- Fixed knots (or break points):
  - **gamlss**: `bs()` and `ns()`
  - **gamlss.util** `fitFicBP()`
- Free knots (or break points):
  - **gamlss.util**: `fitFreeknots()`
  - **gamlss.ad**: `fk()`

Penalised Regression Splines

- Penalised discrete smoothing (smoothing the mean)
- Penalised Beta spline smoothing (smoothing a function)

Penalised discrete smoothing

- data series y_i for $i = 1, 2, \ldots, n$
- Wanted: a smooth series μ
- Two (conflicting) goals: fidelity to y and smoothness
- Fidelity, sum of squares: $S = \sum_i (y_i - \mu_i)^2$
- How to quantify smoothness?
- Use roughness: $R = \sum_i (\mu_i - \mu_{i-1})^2$
- Simplification of Whittaker (1923) "graduation"

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Penalised least squares

- Combine fidelity and roughness

$$Q = S + \lambda R = \sum_i (y_i - \mu_i)^2 + \lambda \sum_i (\mu_i - \mu_{i-1})^2$$

- Parameter λ sets the balance
- Operator notation $\Delta \mu_i = \mu_i - \mu_{i-1}$

$$Q = \sum_i (y_i - \mu_i)^2 + \lambda \sum_i (\Delta \mu_i)^2$$

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Matrix notation

- Penalised least squares objective function

$$Q = ||\mathbf{y} - \boldsymbol{\mu}||^2 + \lambda||\mathbf{D}\boldsymbol{\mu}||^2$$

- Differencing matrix $\mathbf{D}$, such that $\mathbf{D}\boldsymbol{\mu} = \Delta\boldsymbol{\mu}$

$$\mathbf{D} = \begin{bmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix}$$

- Explicit solution

$$\hat{\boldsymbol{\mu}} = \left(I + \lambda\mathbf{D}^\top\mathbf{D}\right)^{-1}\mathbf{y}$$

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Higher order penalties

- Higher order differences are easily defined
- Second order: $\Delta^2\mu_i = \Delta\left(\Delta\mu_i\right) = (\mu_i - \mu_{i-1}) - (\mu_{i-1} - \mu_{i-2})$
- Second order differencing matrix $\mathbf{D}$,

$$\mathbf{D} = \begin{bmatrix} -1 & -2 & 1 & 0 & 0 \\ 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 1 & -2 & 1 \end{bmatrix}$$

- Higher orders are straightforward item in R: `D=diff(diag(M), diff=d)`

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Penalised discrete smoothing

demo.discreteSmo(): for penalised discrete smoothing
demo.interpolateSmo(): for showing the effect of interpolation and extrapolation

Interpolation and extrapolation

- Interpolation is by polynomial of degree $2d - 1$
- Extrapolation: introduce "missing" data in the end(s)
- Extrapolation is by polynomial of degree $d - 1$

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Optimal smoothing

- How much we should smooth
- Let data decide
- Cross-validation, Generalized cross validation, AIC, SBC, (BIC), local maximum likelihood
- Essentially we measure prediction performance

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Penalised Beta spline smoothing: P-splines

- do regression on (cubic) B-splines
- take a large number of them
- put a difference penalty (order 2 or 3) on the coefficients
- tune smoothness with λ
- don't try tp optimize the number of B-spline knots
- relatively small system of equations $(10, 20, 50)$

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Technical details of P-splines

- minimize (with basis $\mathbf{B}$)

$$Q = ||\mathbf{y} - \mathbf{B}\gamma||^2 + \lambda||\mathbf{D}\gamma||^2$$

- explicit solution $\hat{\gamma} = \left(\mathbf{B}^\top\mathbf{B} + \lambda\mathbf{D}^\top\mathbf{D}\right)^{-1}\mathbf{B}\mathbf{y}$
- hat matrix $\mathbf{H} = \mathbf{B}\left(\mathbf{B}^\top\mathbf{B} + \lambda\mathbf{D}^\top\mathbf{D}\right)^{-1}\mathbf{B}^\top$

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P-splines

demo.PenSplines(): for penalised beta splines

The smoothers in gamlss

`pb()`	Penalised beta splines
`cy()`	Cycle Penalised beta splines
`pvc()`	Penalised Varying coefficient
`cs()`	Cubic spline
`lo()`	`loess`
`ga()`	Simon Wood's smoothers from package `mgcv`

The rent data

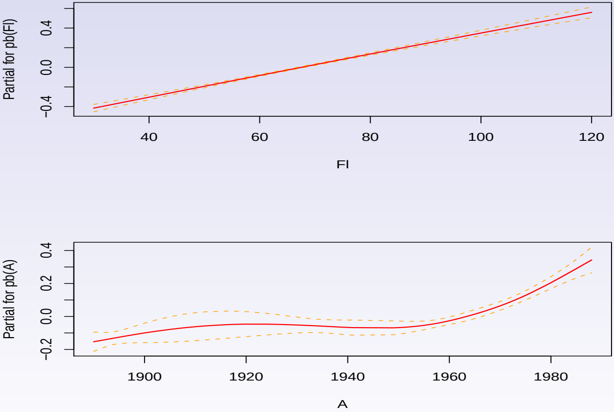
R data file: `rent` in package **gamlss** of dimensions 1969×9 originally given by Fahrmeir *et al.* (1994, 1995).

- `R` : rent response variable, the monthly net rent in DM
- `Fl` : floor space in square meters
- `A` : year of construction
- `SP` : a two level factor for location (whether above average)
- `SM` : a two level factor for location (whether below average
- `B` : a two level factor for bathroom
- `H` : a two level factor for central heating
- `L` : a two level factor for kitchen equipment

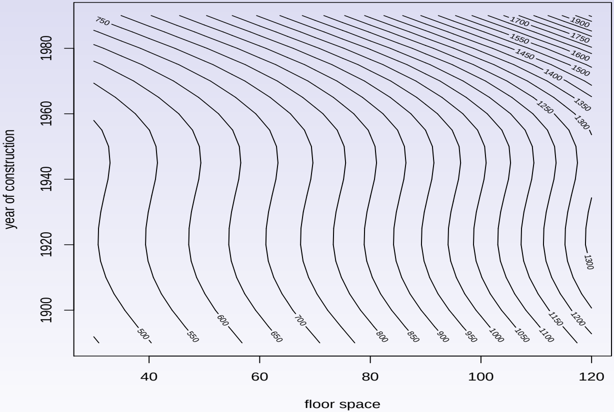
P-splines, the pb() function: additive fit

```
 r1<-gamlss(R~pb(Fl)+pb(A),data=rent, family=GA)
GAMLSS-RS iteration 1: Global Deviance = 27923.61
GAMLSS-RS iteration 2: Global Deviance = 27923.61
op <- par(mfrow = c(2, 1))
term.plot(r1, se = TRUE, partial = TRUE)
par(op)
```

P-splines, the pb() function: term.plot()



P-splines, the pb() function: contour



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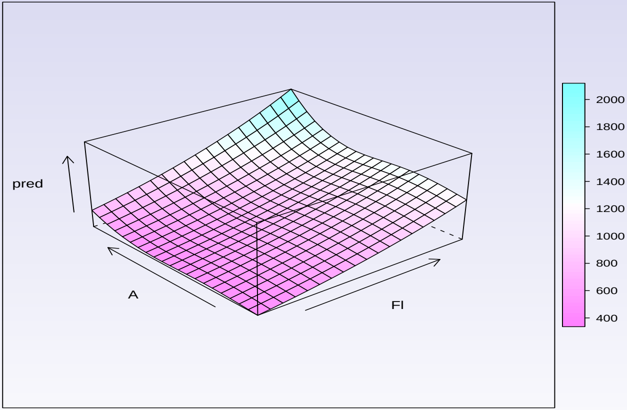
P-splines, the pb() wireframe()



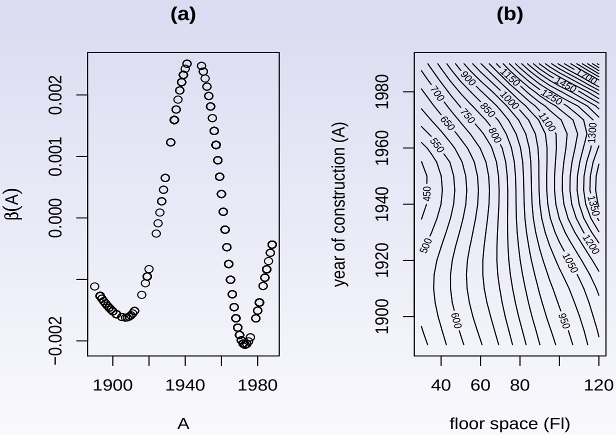
Varying coefficient, the pvc() function

Varying coefficient is a special type of interaction between explanatory variables.
This interaction takes the form of $\beta(R)X$,

Varying coefficient, the pvc() function

```
 r2 <- gamlss(R ~ pb(Fl) + pb(A) + pvc(A, by = Fl),
              data = rent,family = GA)
GAMLSS-RS iteration 1: Global Deviance = 27905.56
GAMLSS-RS iteration 2: Global Deviance = 27905.55
GAMLSS-RS iteration 3: Global Deviance = 27905.56
GAMLSS-RS iteration 4: Global Deviance = 27905.56
GAMLSS-RS iteration 5: Global Deviance = 27905.56
 AIC(r1, r2)
          df      AIC
r2 10.903716 27927.37
r1  7.371373 27938.35
```

The pvc() function



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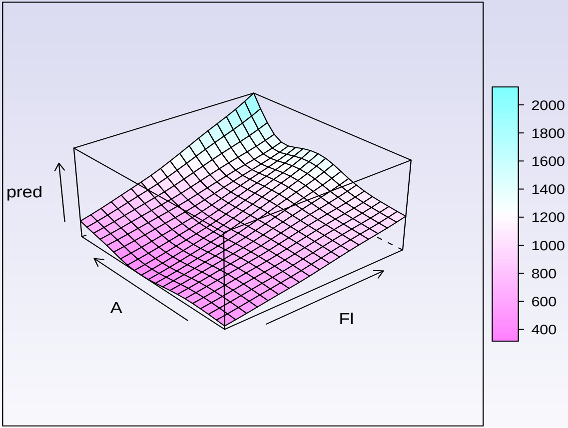
The pvc() function



The loess function lo()

span : how big is the neighbourhood
degree : the degree of the local polynomial

loess, the rent data analysis

```
r11 <- gamlss(R ~ lo(Fl) + lo(A), data = rent, family = GA)
r12 <- gamlss(R ~ lo(Fl, degree = 2) + lo(A, degree = 2),
        data = rent, family = GA)
r21 <- gamlss(R ~ lo(Fl, A, degree = 1), data = rent,
         family = GA)
r22 <- gamlss(R ~ lo(Fl, A, degree = 2), data = rent,
          family = GA)
GAIC(r11, r12, r21, r22, k = 2.5)
          df      AIC
r22 19.08488 27936.78
r21 11.33833 27937.33
r11 10.00128 27946.69
r12 14.12759 27952.95
```

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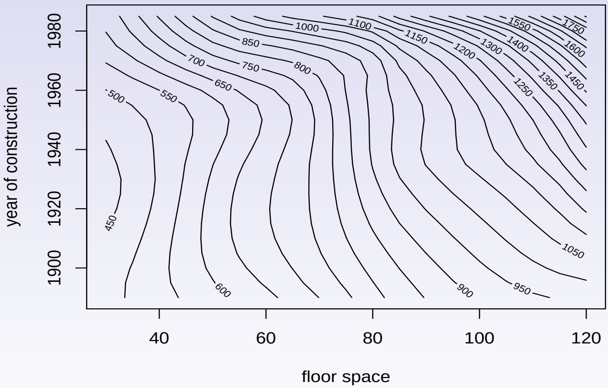
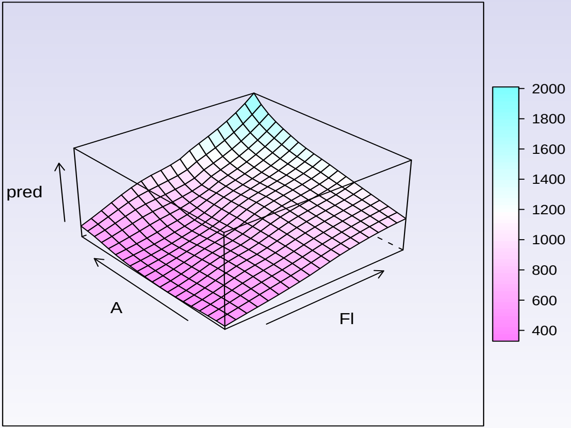
loess, the rent data analysis

```
r223 <- gamlss(R ~ lo(Fl, A, degree = 2, span = 0.3),
   data = rent, family = GA)
r224 <- gamlss(R ~ lo(Fl, A, degree = 2, span = 0.4),
   data = rent, family = GA)
r226 <- gamlss(R ~ lo(Fl, A, degree = 2, span = 0.6),
   data = rent, family = GA)
AIC(r22, r223, r224, r226, k = 2.5)
           df      AIC
r22  19.08488 27936.78
r226 16.29977 27938.16
r224 22.46753 27939.15
r223 28.89809 27946.83
```

loess, the rent data analysis



loess, the rent data analysis



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The ga() function

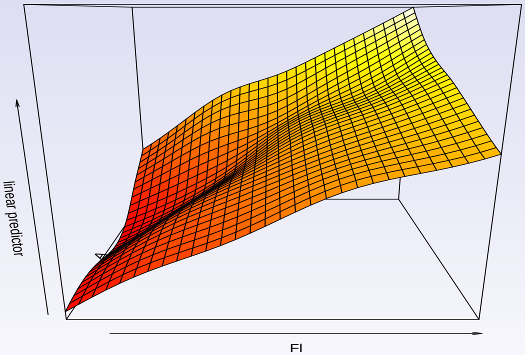
Interface with the `gam()` function of package `mgcv`

For example `ga(~(x1,x2)+ s(x3))`

The ga() function

```
library(gamlss.add)
g1 <- gamlss(R ~ ga(~s(Fl, A) ), data = rent, family = GA)
vis.gam(g1$mu.coefSmo[[1]])
```

ga(), the rent data analysis



END

for more information see

www.gamlss.org